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A GEOMETRIC PROPERTY OF TOPOLOGICAL AMENABLE SEMIGROUPS

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Abstract

Let S be a locally compact semigroup, M(S) the Banach algebra of all bounded regular Borel measures on S and $M(S)^*$ its continuous dual. Let $M_0(S)$ be the set of probability measures in M(S), $N(S)=\{F\epsilon M(S)^*: \inf\{\|\mu\odot F\|:\mu\epsilon M_0(S)\}=0\}$ and $\mathcal{H}=\{H\epsilon M(S)^*:H=\sum_{k=1}^{n}(F_k-\mu_k\odot F_k)$, for some $F_1,\ldots, F_n\,\epsilon\, M(S)^*$ and $\mu_1,\ldots,\mu_n\,\epsilon\, M_0(S)$. A geometric property of topological left amenable semigroups is proved and as an application it is shown that S is toplogical left amenable if and only if $N(S)=\overline{\mathcal{H}}$.

1. Definitions and Notations: Let S be a locally compact semigroup with convolution measure algebra M(S). Let $M(S)^*$ be the continuous dual of $M(S)^*$ and 1 the linear functional in $M(S)^*$ such that $1(\mu)=\mu(S)$, for all μ in M(S). For each $\mu\,\epsilon\, M(S)$ and each $F\,\epsilon\, M(S)^*$, define $L\mu F$ by $(L\mu F)(\nu)=F(\mu*\nu)$, $\nu\epsilon M(S)$. Let $M_0(S)=\{\mu\epsilon M(S): \mu\geqslant 0$ and $\|\mu\|=1\}$. An element $M\,\epsilon\, M(S)^{**}$ is called a mean if,

$$\inf\{F(\mu):\mu\epsilon M_0(S)\}\leqslant M(F)\leqslant \sup\{F(\mu):\mu\epsilon M_0(S)\},$$

for all $F\,\epsilon\, M(S)^*$. A mean M is called topological left invariant if $M(L\mu F)=M(F)$, for any $F\,\epsilon\, M(S)^*$ and $\mu\epsilon M_0(S)$. If there is a toplogical left invariant mean on $M(S)^*$ we say that S is toplogical left amenable.

2. Basic Results: Let X be a normed linear space. An anti-action of M(S) on X is a bilinear mapping
$T: M(S)\times X\longrightarrow X$ denoted by $(\mu, x)\longrightarrow T_\mu x$ such that

(i) $T\mu$ is bounded for all $\mu\epsilon M(S)$.

(ii) $T\mu.\nu = T\,\nu O\, T\,\mu$ for all $\mu,\nu\epsilon M(S)$.

Let $O(x)=\{T_\mu x:\mu\epsilon M_0(S)\}$ be the orbit of x and K_X the linear span of $\{x-T_\mu x: x\epsilon X, \mu\epsilon M_0(S)\}$.

We now prove the following theorem which is toplogical analogue of [2, Theorem 5 part I(a)].

Theorem 2.1. Let T be an anti-action of M(S) on a normed linear space X such that

(i) $\|T_\mu\|\leqslant 1$ for all $\mu\epsilon M_0(S)$

(ii) For each $x\,\epsilon\, X$, the map $\mu\longrightarrow T_\mu x$ from M(S) into X is norm-norm continuous.

If S is topological left amenable then $\operatorname{dis}(0, O(x)) = \operatorname{dis}(x, K_X)$ for all $x\,\epsilon\, X$, where $\operatorname{dis}(x, A)=\inf\{\|x-y\|:y\epsilon A\}$ for every $A\subseteq X$.

PROOF: Let $\{\mu_\alpha\}$ be a net in $M_0(S)$ converging to topological left invariance in norm. That is, $\|\mu*\mu_\alpha-\mu_\alpha\|\to 0$ for any $\mu\epsilon M_0(S)$. (See [4, Therorem 3.1, (3) $\Leftrightarrow$(4)]). Then

$$\|T_{\mu_\alpha}(x-T_\mu x)\|=\|T_{\mu_\alpha}x-T_{\mu*\mu_\alpha}x\|=\|T_{\mu_\alpha-\mu*\mu_\alpha}x\|\to 0.$$

for all $x\,\epsilon\, X$ and $\mu\epsilon M_0(S)$ by linearity and norm-norm continuity of the map $\mu\to T_\mu x$. Thus $\|T_{\mu_\alpha}y\|\to 0$ for all $y\epsilon K_X$.

Let $z\epsilon X, \varepsilon>0$ be given. There is $h\,\epsilon K_X$ such that $\|z+h\|<\operatorname{dis}(z,K_X)+\varepsilon$. Since $\|T_{\mu_\alpha}h\|\to 0$, there is some α_0 such that $\|T_{\mu_{\alpha_0}}h\|<\varepsilon$. Hence:

$$\|T_{\mu_{\alpha_0}}z\|\leqslant\|T_{\mu_{\alpha_0}}(z+h)\|+\|T_{\mu_{\alpha_0}}h\|$$
$$\leqslant\|z+h\|+\varepsilon<\operatorname{dis}(z,K_X)+2\varepsilon.$$

Thus $\operatorname{dis}(0, O(z))\leqslant \operatorname{dis}(z, K_X)$. The converse inequality is obvious, since clearly $O(x)\subseteq x+K_X$ for each $x\,\epsilon\, X$. Thus: $\operatorname{dis}(0,O(x))\geqslant\operatorname{dis}(0,x+K_X)=\operatorname{dis}(x,K_X)$, and the proof is complete.

REMARK: For discrete semigroups the above theorem can be found in [1, p. 99-104]. The proof here is different but analogous to that in Granirer [2] who uses the term antirepresentation instead of anti-action and considers only antirepresentations of the semigroups S instead of the measure algebra M(S).

Let $\mathcal{H}=\{H\epsilon M(S)^*:H=\sum_{k=1}^{n}(F_k-\mu_k\odot F_k)$, for some $F_1,\ldots,F_n\,\epsilon\, M(S)^*$ and $\mu_1,\ldots\mu_n\,\epsilon\, M_0(S)\}$ and $N(S)=\{F\epsilon M(S)^*: \inf\{\|\mu\odot F\|:\mu\epsilon M_0(S)\}=0\}$. As an application of Theorem 2.1. we now prove the following theorem.

Theorem 2.2 S is toplogical left amenable if and only if $N(S)=\overline{\mathcal{H}}$, where $\overline{\mathcal{H}}$ denotes the closure of $\mathcal{H}$ in the norm topology of $M(S)^*$.

PROOF: First note that if we take X to be $M(S)^*$ and T to be the anti-action $\{L_\mu:\mu\epsilon M(S)\}$ over $M(S)^*$ then $K_{M(S)^*}=\mathcal{H}$. Now, if $M(S)^*$ has a topological left invariant mean then by Theorem 2.1, $\operatorname{dis}(0, O(F))=\operatorname{dis}(F,\mathcal{H})$ for all $F\,\epsilon\, M(S)^*$. Therefore $N(S)=\overline{\mathcal{H}}$. Conversely if $N(S)=\overline{\mathcal{H}}$, then since $\mathcal{H}$

is closed under addition we conclude that N(S) is also closed under addition, hence by [3, Theorem 2.3], S is topological left amenable.

REMARK: There is another proof for the necessity part of Theorem 2.2. as follows.

If $M(S)^*$ has a topological left invariant mean, we say that a functional F in $M(S)^*$ is topological left almost convergent to the constant ß if M(F) = ß for every topological left invariant mean M on $M(S)^*$. It can be proved, following the idea in [5, § 7.] that F is topological left almost convergent to ß if and only if the constant functional ß.1 is in the norm closure of the convex set $M_0(S)\,.\odot F$ and that $\mathcal{H}$ is precisely the set of all functionals F in $M(S)^*$ which are topological left almost convergent to 0, where the closure is taken in the norm topology of $M(S)^*$. Consequently, $N(S) = \{ F \in M(S)^*:$ 0 is in the norm closure of $M_0(S).\odot F\} = \overline{\mathcal{H}}$.

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